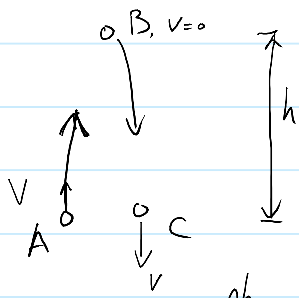
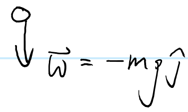
Potential energy and energy conservation

motivation: Consider



only force acting on the ball is the weight, mg.



From A to B, $W_{tot} = \int_0^h (-mg\hat{j})\cdot dy\,\hat{j} = -mgh < 0$

"mgh" amount of energy is taken away from object by gravity.

From B to C, $W_{tot} = \int_h^0 (-mg\hat{j})\cdot dy\,\hat{j} = -mg(0-h) = mgh > 0$

"mgh" amount of energy is transferred to object by gravity.

Conclusion:

- Gravity between the Earth and the object could take away and give energy to the object depending only on the position between the Earth and the object, h.
- If the object goes up and then return to its original position, the gravity first take away then return exactly the same amount of energy, like an energy storage.

Let's define $\Delta U(y_i, y_f) = -W_{g \text{ from } y_i \text{ to } y_f}$. depends only on initial and final positions.

$$\Rightarrow \quad W_{tot} = \Delta K$$

$$\Rightarrow \quad W_g = \Delta K$$

$$\Rightarrow \quad 0 = \Delta K - W_g = \Delta K + \Delta U$$ for all processes. $(A \to B,\ B \to C)$

Define $E_{tot} = K + U$

total mechanical energy

$\Delta E_{tot} = 0$ conservation of mechanical energy.

In general, we define a potential energy only when the corresponding force/interaction return the same amount of energy if the object move back to it original position.

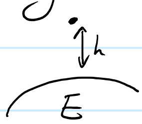
we call these forces conservative force.

Definition: Change in potential energy

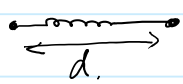
$$\boxed{\Delta U_{A\to B} = U_B - U_A = -W^{\text{conservative}}_{A\to B} = -\int_A^B \vec{F}_c \cdot d\vec{s}}$$

Properties of ΔU: (i) due to interaction, e.g. gravity, spring, Coulomb's force...
an isolated particle $\Rightarrow$ No U.

(ii) Potential Energy depends only on relative position of the objects in the system.

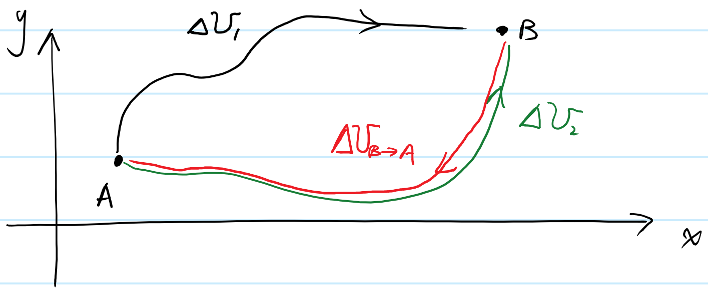


or



(iii) ΔU depends only on initial (A) and final (B) positions.
The path going from A to B is not important.

$\underline{\Delta U \text{ is path independent quantity.}}$



$\Delta U_1 = \Delta U_2$

(iv) Interchanging the initial and final position:

$$\boxed{\Delta U_{B\to A} = -\Delta U_{A\to B}} \overset{\text{in above example}}{=} -\Delta U_2$$

$\Rightarrow$ the path $A \to B \to A$

$$\Delta U_{A\to B\to A} = \Delta U_1 + \Delta U_{B\to A}$$
$$= \Delta U_1 - \Delta U_2$$
$$= 0$$

$\Rightarrow$ For any closed path: $\boxed{\Delta U_{\substack{\text{closed}\\\text{path}}} = 0}$

(v) It is convenient to define potential energy at a point (x)
even potential energy is originally defined as a change (ΔU)

we write:

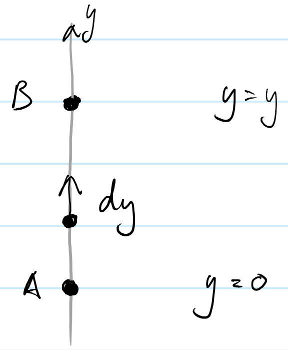
$$U(x) = \underset{\uparrow}{U_0} + \Delta U_{x_0 \to x}$$

U at the reference (x_0)

$$\boxed{U(x) = U_0 - \int_{x_0}^{x} \vec{F}\cdot d\vec{s}}$$

U_0 can be freely chosen.

Example. Gravity on Earth. $\vec{F} = -mg\,\hat{j}$



$$\Delta U_{A\to B} = -\int_0^y \underbrace{(-mg\,\hat{j})}_{\vec{F}} \cdot \underbrace{(dy'\,\hat{j})}_{d\vec{s}}$$

$$= mg\int_0^y dy'$$

$$= mgy$$

$\Rightarrow$ Potential Energy at B $= U(y) = U_A + \Delta U_{A\to B}$

$$= U_A + mgy$$

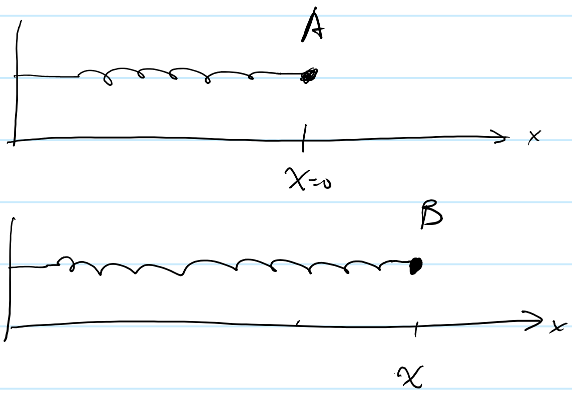
now we choose the potential energy at A to be zero

i.e. choose $U_A = 0$

$\Rightarrow$ $\underline{U(y) = mgy \quad \text{given} \quad U_A = U(y=0) = 0}$

This defines $U(y)$ at any y completely.

Example: Spring



$$\vec{F}_{spring} = -kx\,\hat{i}$$

$$\Delta U_{A\to B} = -\int_{\vec{r}_A}^{\vec{r}_B} \vec{F}_{spring} \cdot d\vec{s}$$

$$= -\int_0^x -kx'\,\hat{i} \cdot dx'\,\hat{i}$$

$$= k\int_0^x x'\,dx'$$

$$= \frac{1}{2}kx^2$$

$\Rightarrow$ $U(x) = U_A + \frac{1}{2}kx^2$

choose $U_A = 0$, $\boxed{U(x) = \frac{1}{2}kx^2}$

what if the spring is compressed?

Try to take $x_B = -d$ (d is the compressed length.)

The result will be $U = \frac{1}{2}kd^2$

- Compressing and stretching a spring with the same distance require (store) the same amount of energy.

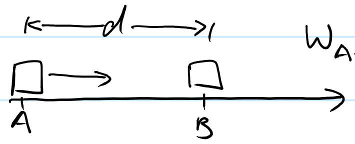
Non-conservative force

Work done by non-conservative force is <u>path dependent.</u>

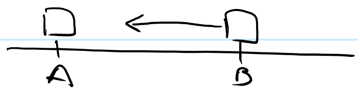
$$W^{①}_{A\to B} \neq W^{②}_{A\to B} \iff W_{\substack{\text{closed}\\ \text{path}}} \neq 0 \quad \forall \text{ path.}$$

$\Rightarrow$ cannot define potential energy.

e.g. friction



$W_{A\to B} = -f_k \cdot d$



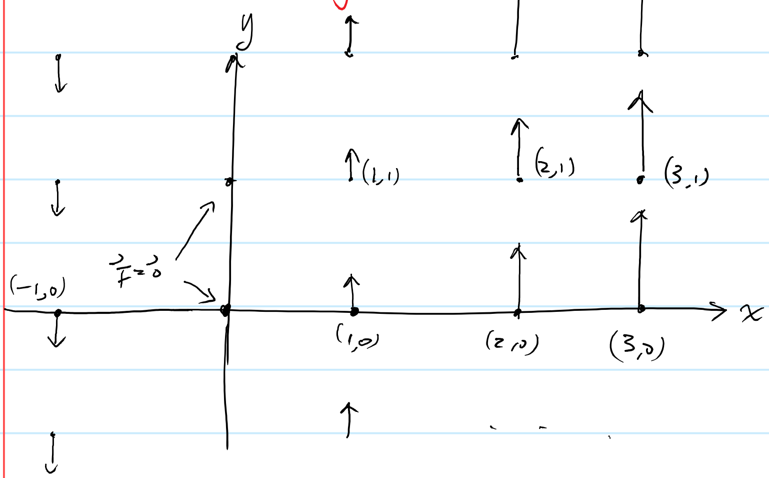
$W_{B\to A} = -f_k d$

$W_{A\to B\to A} \neq 0$

Example To prove a force is non-conservative, one needs to show the work done by the force along a closed path to be non-zero.

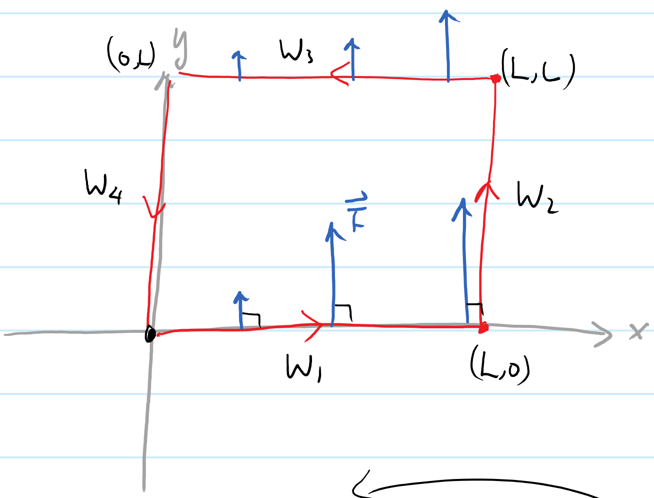
Consider $\vec{F}(x,y) = C x \hat{j}$ where C is a constant.

force varies with location.



Consider a closed path

$(0,0) \to (L,0) \to (L,L) \to (0,L) \to (0,0)$



$$W_1 = \int_{(0,0)}^{(L,0)} \vec{F} \cdot d\vec{s}$$

$$= C \int_0^L x \hat{j} \cdot dx \hat{i} = 0 \quad \because \hat{j} \cdot \hat{i} = 0$$

$$W_2 = \int_{(L,0)}^{(L,L)} \vec{F} \cdot d\vec{s} = C \int_{y=0}^{y=L} x \hat{j} \cdot dy \hat{j} \qquad \underline{\underline{x = L}}$$

$$= C L \int_0^L dy = C L^2$$

$W_3 = 0$ again $\vec{F} \cdot dx \hat{i} = 0$

$W_4 = 0 \quad \because \vec{F} = \vec{0}$ along y-axis $(x=0)$

$$W_{\text{closed path}} = W_1 + W_2 + W_3 + W_4$$
$$= C L^2 \neq 0$$

$\Rightarrow$ $\underline{\vec{F}(x,y) = C x \hat{j} \text{ is non-conservative}}$

Work-energy theorem (revised)

$$W_{tot} = \Delta K$$

$$W_c + W_{NC} = \Delta K$$

conservative non-conservative

def.

$$W_{NC} = \Delta K - W_c \equiv \Delta K + \Delta U$$

$$\Rightarrow \boxed{W_{N.C.} = \Delta K + \Delta U \equiv \Delta E_{tot}}$$

or W_{other}

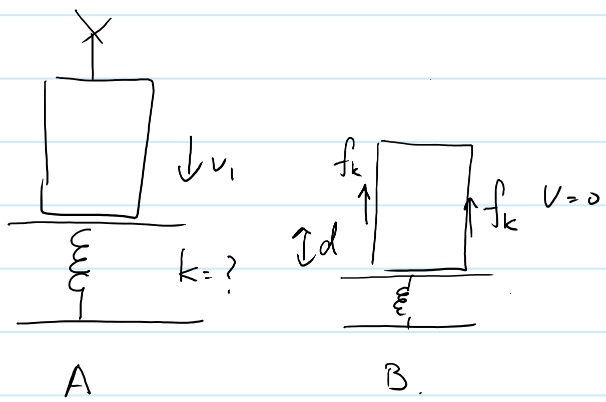
$$W_{N.C.} = 0 \iff \Delta E_{tot} = 0$$

total mechanical energy is conserved

$$\Downarrow$$

$$K_i + U_i = K_f + U_f.$$

Application of work-energy theorem with potential energy



$$W_{N.C} = -f_k d$$

$$\Delta K = 0 - \frac{1}{2} m v_1^2$$

$$\Delta U = \Delta U_{spring} + \Delta U_g$$

$$\Delta U_{spring} = \frac{1}{2} k d^2 - 0$$

$$\Delta U_g = 0 - mgd$$

$$W_{N.C.} = \Delta K + \Delta U$$

$$\Rightarrow \quad -f_k\, d = -\frac{1}{2} m v_1^2 + \frac{1}{2} k d^2 - mgd$$

$$k = \frac{m v_1^2 + 2mgd - 2 f_k d}{d^2}$$

<u>Finding $\vec{F}$ from U.</u>

$$\begin{tikzcd} \vec{F} \arrow[r, bend left, "\int \vec{F}\cdot d\vec{s}"] & U \arrow[l, bend left, "\frac{dU}{dx}"] \end{tikzcd}$$

in 1D. $\quad \Delta U = -W = -F \cdot \Delta x \quad$ for F is constant

$$\Rightarrow \quad F = -\frac{\Delta U}{\Delta x} \longrightarrow \boxed{F = -\frac{dU}{dx}}$$

simple check: $\quad U(y) = mgy$

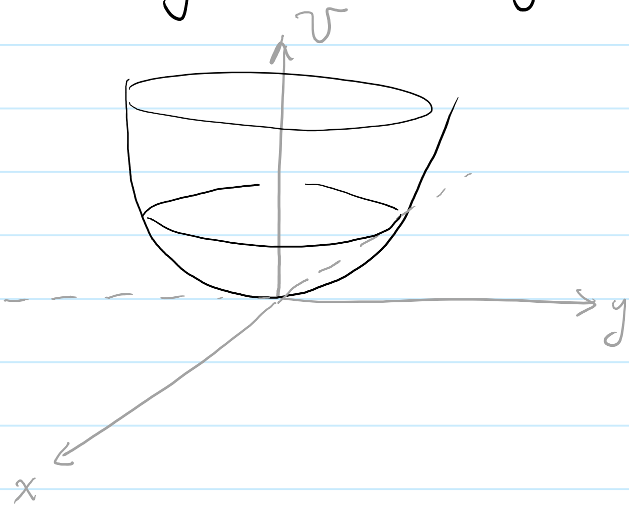
$$-\frac{dU}{dy} = -mg = F \text{ of gravity on Earth.}$$

in 3D, $\quad U = U(x,y,z) \Rightarrow \vec{F} = F_x \hat{i} + F_y \hat{j} + F_z \hat{k}$

$$\begin{cases} F_x = -\dfrac{\partial U}{\partial x} \\[2mm] F_y = -\dfrac{\partial U}{\partial y} \\[2mm] F_z = -\dfrac{\partial U}{\partial z} \end{cases}$$

$\leftarrow$ partial derivative, take derivative w.r.t. x while keeping y and z as constants.

Example given $U = U(x,y) = \frac{1}{2}k(x^2+y^2)$



$$F_x = -\frac{\partial U}{\partial x} = -\frac{\partial}{\partial x}\left[\frac{1}{2}k(x^2+y^2)\right]$$

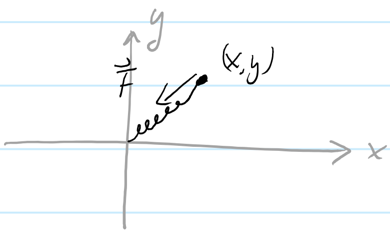
$$= -\frac{1}{2}k \frac{\partial}{\partial x}(x^2+y^2)$$

$$= -\frac{1}{2}k\left(\frac{\partial x^2}{\partial x} + \frac{\partial y^2}{\partial x}\right)$$

treat y as constant.

$$= -\frac{1}{2}k(2x + 0)$$

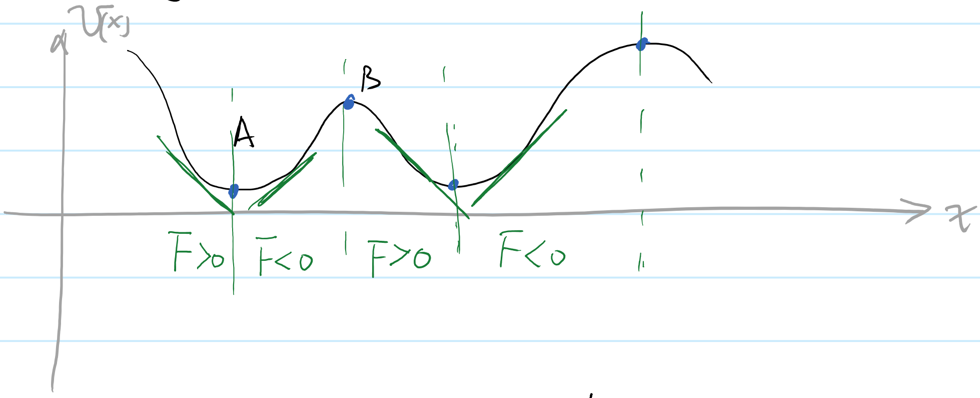
$$= -kx$$



$\vec{F} = -k\vec{r}$

Similarly, $F_y = -ky$.

Potential Energy Graph



At $\bullet$s slope $= 0 \Rightarrow \frac{dU}{dx} = 0 \Rightarrow F = 0$

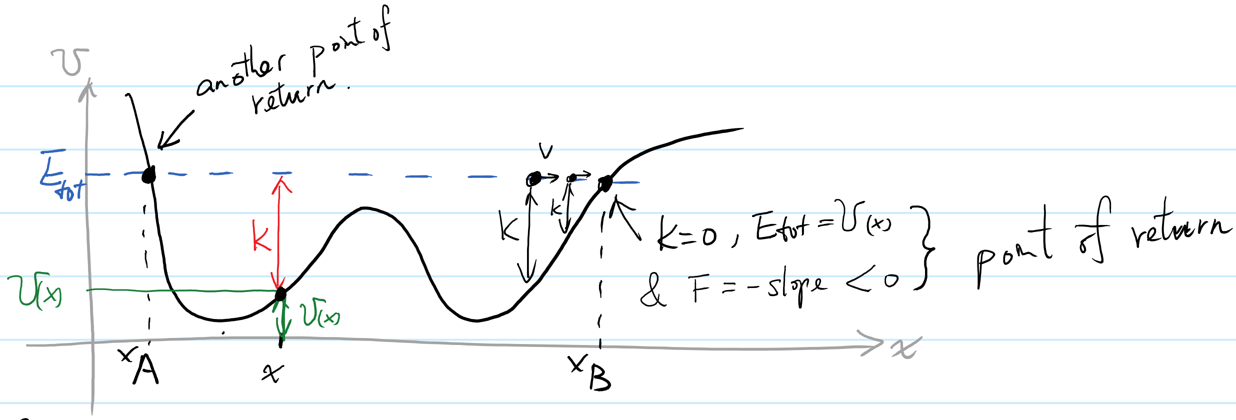
They are the equilibrium points.

If a particle move slightly away from A, either left or right, it will be pushed by the force back to A.

A is called a **stable** equilibrium.

If a particle move slightly away from B, either left or right, it will be pushed by the force away from B.

B is called an **unstable** equilibrium.



Given a total energy E_{tot}, say $E_{tot} = 10\,J$

at any point x, the potential energy is $U(x)$

Since $E_{tot} = K + U(x)$ at any point

$$K = E_{tot} - U(x)$$

The above diagram shows a particle with total energy E_{tot} will bounces back and forth between x_A and x_B.